US system: lb (force), slug (mass) ×32; (12) ft
metric system: Newtons (force), kg (mass) ×9.8; (100) m

5.1 - Linear Models: Initial-Value Problems

Undamped motion: Down is positive

Example: A mass weighing 32 pounds stretches a spring 2 feet. Determine the amplitude and period of motion if the mass is initially released from a point 1 foot above the equilibrium position with an upward velocity of 2 □/s. How many complete cycles will the mass have completed at the end of 4π seconds?

$x(0) = -1$

$x'(0) = -2$

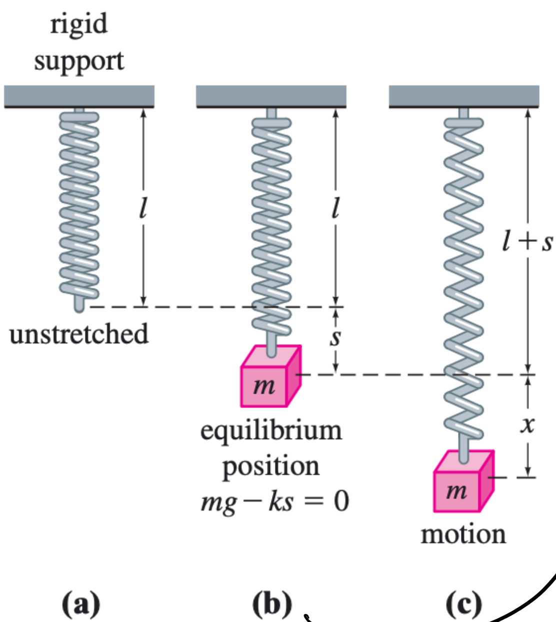


FIGURE 5.1.1 Spring/mass system

Hooke's Law:

$|F| = k|s|$ (spring constant) $F = 32$ lb, $|s| = 2$ ft

$32 = k(2) \Rightarrow k = 16$ lb/ft

(b) $mg = ks \Rightarrow mg - ks = 0$

mg: force due to gravity and mass; ks: restoring force of spring

Assuming down is positive, if the mass is moving, we have $F = ma$ with $a = \frac{d^2x}{dt^2}$

so $m\frac{d^2x}{dt^2} = -k(x+s) + mg = -kx - ks + mg$

$$m\frac{d^2x}{dt^2} + kx = 0$$

Dividing by m and letting $\omega^2 = \frac{k}{m}$ yields

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \Rightarrow$$ Aux eqn: $m^2 + \omega^2 = 0$

$m = \pm \omega i$

$$\boxed{x(t) = c_1 \cos \omega t + c_2 \sin \omega t}$$

$F = ma$

$\omega = \sqrt{k/m}$ $\quad k = 16 \quad$ mass: $32\,lb = \text{slugs}(32)$

mass $\quad m = 1 \Rightarrow k/m = 16 \Rightarrow \omega = 4$

$x(t) = c_1 \cos 4t + c_2 \sin 4t$

$x(0) = -1 \Rightarrow c_1 = -1$

$x(t) = -\cos 4t + c_2 \sin 4t$

$x'(t) = 4 \sin 4t + 4c_2 \cos 4t \qquad x'(0) = -2$

$c_2 = -\frac{1}{2} \Rightarrow x(t) = -\cos 4t - \frac{1}{2}\sin 4t$

period: $T = \frac{2\pi}{\omega} = \frac{2\pi}{4} = \frac{\pi}{2}$ sec ✓

$\frac{1}{\text{per}}$ = freq: $\frac{2}{\pi}$ cycles/sec (Hz)

4π sec: $\frac{2}{\pi}(4\pi) = 8$ cycles ✓

For the amplitude . . .

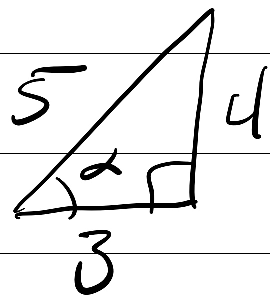
$y = a\sin(\omega x)$

per: $\frac{2\pi}{\omega}$

ϕ' : $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

Side bar: Recall $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

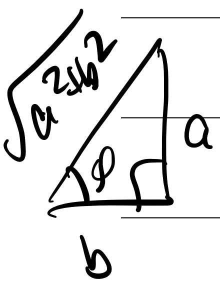
So consider $4\cos\beta + 3\sin\beta$



$$5\left(\frac{4}{5}\cos\beta + \frac{3}{5}\sin\beta\right)$$

so $\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$

In general, given $a\cos\theta + b\sin\theta$



$$\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\cos\theta + \frac{b}{\sqrt{a^2+b^2}}\sin\theta\right)$$

Suppose $\frac{a}{\sqrt{a^2+b^2}} = \sin\phi$, $\frac{b}{\sqrt{a^2+b^2}} = \cos\phi$)

Then we have $\sqrt{a^2+b^2}\,\sin(\theta+\phi) = a\cos\theta + b\sin\theta$

$\Rightarrow \tan\phi = \frac{a}{b}$

$c_1 = -1$ $\quad c_2 = -\frac{1}{2}$

Back to $x(t) = -\cos 4t - \frac{1}{2}\sin 4t$

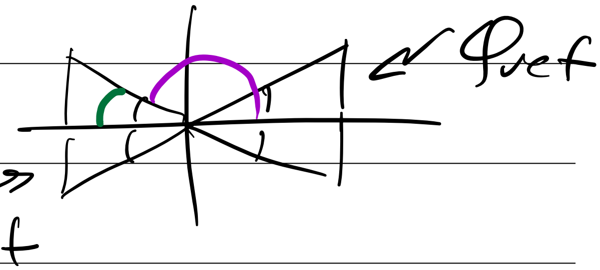
$c_1 = -1$, $c_2 = -\frac{1}{2}$ (Q III)

$\sqrt{c_1^2 + c_2^2} = \sqrt{1+\frac{1}{4}} = \frac{\sqrt{5}}{2}$

$$x(t) = \frac{\sqrt{5}}{2}\left(-\frac{1}{\sqrt{5}/2}\cos 4t\right) + \left(-\frac{1}{\sqrt{5}}\sin 4t\right)$$

$\tan\phi = \frac{c_1}{c_2} \Rightarrow \tan\phi = 2$ (in Q III)

$\phi_{ref} \approx 1.107$



$\phi_{actual} \approx 1.107 + \pi \approx 4.248$

$x(t) = \frac{\sqrt{5}}{2} \sin(4t + 4.248)$

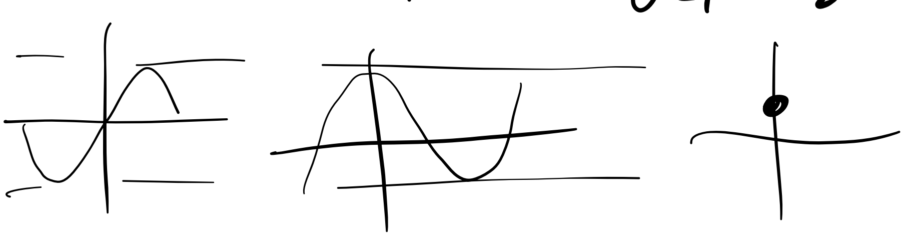
$\sqrt{c_1^2 + c_2^2}$ ω $\phi: \tan\phi = \frac{c_1}{c_2}$

amp: $\frac{\sqrt{5}}{2}$ ft

To generalize, $x(t) = c_1 \cos\omega t + c_2 \sin\omega t$ has period $\frac{2\pi}{\omega}$, freq $= \frac{1}{per} = \frac{\omega}{2\pi}$, phase shift $-\phi$, where $\tan\phi = \frac{c_1}{c_2}$, and amplitude $\sqrt{c_1^2 + c_2^2}$.

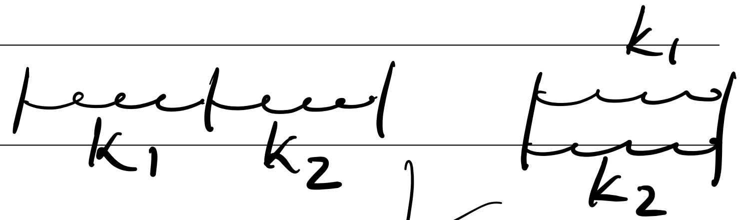
and we get $x(t) = \sqrt{c_1^2 + c_2^2} \sin(\omega t + \phi)$

OR $x(t) = \sqrt{c_1^2 + c_2^2} \cos(\omega t - \phi')$



$\tan\phi' = \frac{c_2}{c_1}$

multiple springs:



Multiple springs in parallel: $k_{eq} = k_1 + k_2$

Multiple springs in series: $k_{eq} = \frac{k_1 k_2}{k_1 + k_2}$

Damped motion: $m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}$

(beta)

ma ↗ (for $m\frac{d^2x}{dt^2}$)

restoring force from spring ↗ (for $-kx$)

damping force is proportional to instantaneous velocity (for $\beta\frac{dx}{dt}$)

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

Let $2\lambda = \frac{\beta}{m}$, $\omega^2 = \frac{k}{m}$

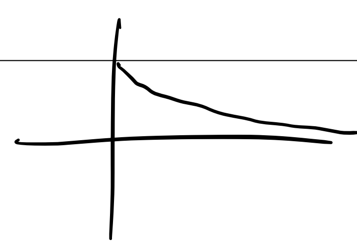
Aux eqn solution: $m = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$

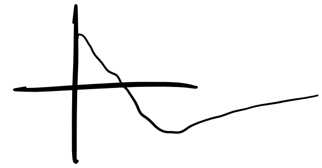
<u>Case I</u>: $\lambda^2 - \omega^2 > 0$

$$x(t) = c_1 e^{(-\lambda + \sqrt{\lambda^2 - \omega^2})t} + c_2 e^{(-\lambda - \sqrt{\lambda^2 - \omega^2})t}$$

This does not oscillate.

This system is <u>overdamped</u>.





Case II: $\lambda^2 - \omega^2 = 0$

$$x(t) = c_1 e^{-\lambda t} + c_2 t e^{-\lambda t}$$

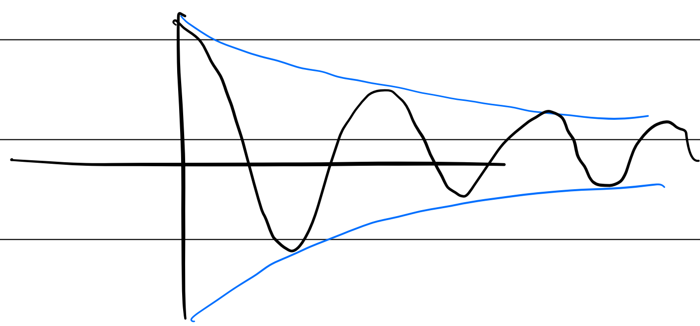
This is <u>critically damped</u>

(not by accident)

Case III: $\lambda^2 - \omega^2 < 0$

$$x(t) = e^{-\lambda t}\left(c_1 \cos(\omega^2 - \lambda^2)t + c_2 \sin(\omega^2 - \lambda^2)t\right)$$

This is <u>underdamped</u>



decreasing w/ variable amplitude

Driven motion: $m\dfrac{d^2x}{dt^2} + kx + \beta\dfrac{dx}{dt} = f(t)$

↑ nonhomogeneous

LRC-series circuit: $L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t)$, which is

$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$

can be either underdamped,
critically damped, or overdamped.

Example: Find the charge on the capacitor in an LRC-series circuit when $L = \frac{1}{4}$ h, $R = 20\ \Omega$, $C = \frac{1}{300}$ f, $E(t) = 0$ V, $q(0) = 4$ C, and $i(0) = 0$ A. Is the charge on the capacitor ever equal to zero?